

# Instanton Solution in Tachyon Cosmology

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We find an exact classical solution in Euclidean gravity coupled to a scalar field with a particular form of potential commonly used in tachyon cosmology. This solution represents a tunneling between two vacua.

## 1. Introduction

There is mounting evidence that string theory has a landscape of vacua [1]. In this picture, it is likely that the universe experiences a tunneling to a nearby vacuum. As a first step, it is important to understand the tunneling process in the low energy gravity description. For instance, it is well known that the false vacuum decay by bubble nucleation is described by the Coleman-De Luccia instanton [2]. However, the instanton solution is known only for a special circumstances where the thin-wall approximation is valid<sup>1</sup>. It is desirable to have an analytic control over the behavior of such instantons.

In this paper, we take a modest step toward this goal. We consider a special form of scalar potential

$$V(\phi) = \frac{1}{\cosh \phi} , \quad (1.1)$$

and find a tunneling solution interpolating two vacua at  $\phi = \pm\infty$ . This particular form of potential naturally appears in the study of cosmology in the presence of unstable D-brane [4,5]. The scalar field  $\phi$  is identified as the open string tachyon  $T$  on the D-brane. In such a scenario, it is usually assumed that the kinetic term of tachyon field is given by the Born-Infeld form. However, this leads to a non-linear dynamics of tachyon field and it is beyond our capability of analytic control. Therefore, for simplicity we assume that the kinetic term of scalar field is canonical.

## 2. Instanton Solution

In this section, we construct an exact solution of the Euclidean gravity coupled to a scalar field with a special choice of the scalar potential  $V(\phi)$ . The Euclidean action of the system is given by

$$S = \int d^4x \sqrt{g} \left( -\frac{R}{16\pi G_N} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right) . \quad (2.1)$$

We look for an  $SO(4)$  symmetric solution with the ansatz

$$ds^2 = d\tau^2 + a(\tau)^2 d\Omega_3^2, \quad \phi = \phi(\tau) , \quad (2.2)$$

where  $d\Omega_3^2$  is the metric of round 3-sphere of unit radius. The equations of motion are

$$\ddot{\phi} + \frac{3\dot{a}}{a} \dot{\phi} - \frac{dV(\phi)}{d\phi} = 0 , \quad (2.3)$$

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<sup>1</sup> See [3] for a study of bubble nucleation in 2d quantum gravity.

and

$$\frac{\dot{a}^2}{a^2} = \frac{1}{a^2} + \frac{2}{3M_{pl}^2} \left( \frac{1}{2}\dot{\phi}^2 - V(\phi) \right) . \quad (2.4)$$

Here we defined the Planck mass  $M_{pl}$  as

$$M_{pl}^2 = \frac{1}{4\pi G_N} . \quad (2.5)$$

To solve (2.3) and (2.4), we follow the strategy of [6,7]. Namely, we further assume that  $\dot{\phi}$  and  $a$  are related by some function  $f(a)$

$$\dot{\phi}^2 = a \frac{df(a)}{da} . \quad (2.6)$$

Multiplying  $\dot{\phi}$  to (2.3) and use (2.6), we can easily see that (2.3) becomes total derivative with respect to  $\tau$

$$\frac{d}{d\tau} \left( \frac{1}{2}\dot{\phi}^2 - V(\phi) + 3f(a) \right) = 0 . \quad (2.7)$$

Integrating this equation and using (2.6), we find

$$V(\phi) = \frac{a}{2} \frac{df(a)}{da} + 3f(a) . \quad (2.8)$$

Here we have included the integration constant in the definition of  $f(a)$ . Plugging (2.8) into (2.4), we get a closed equation for  $a$

$$\dot{a}^2 = 1 - \frac{2}{M_{pl}^2} a^2 f(a) . \quad (2.9)$$

Now we have a freedom to choose  $f(a)$ . Each choice of  $f(a)$  corresponds to a particular form of the scalar potential obtained from (2.8) and (2.6). As discussed in [6], the following choice of  $f(a)$  leads to a solvable system

$$f(a) = \frac{m^2 M_{pl}^2}{2} \left( 1 - \beta^2 + \beta^2 m^2 a^2 \right) . \quad (2.10)$$

Here  $\beta$  is a dimensionless parameter and  $m$  is a mass parameter.

Now, let us first consider the solution for  $a$ . When  $f(a)$  is given by (2.10), (2.9) becomes

$$\dot{a}^2 = (1 - m^2 a^2)(1 + \beta^2 m^2 a^2) . \quad (2.11)$$

This equation is solved by the Jacobi elliptic function  $\text{sn}(z, k)^2$

$$a(\tau) = \frac{1}{m} \text{sn}(m\tau, i\beta) . \quad (2.13)$$

Next consider the solution  $\phi(\tau)$ . Plugging the form of  $f(a)$  into (2.6), we find

$$\dot{\phi}^2 = m^4 M_{pl}^2 \beta^2 a^2 . \quad (2.14)$$

Again, this is solved by a combination of Jacobi elliptic functions  $\text{cn}(z, k)$  and  $\text{dn}(z, k)$

$$\phi(\tau) = M_{pl} \tanh^{-1} \left( \frac{\text{cn}(m\tau, i\beta)}{\text{dn}(m\tau, i\beta)} \right) . \quad (2.15)$$

Using the identity of Jacobi elliptic functions

$$\text{sn}^2(z, k) + \text{cn}^2(z, k) = 1, \quad k^2 \text{sn}^2(z, k) + \text{dn}^2(z, k) = 1, \quad (2.16)$$

one can easily see that  $a(\tau)$  and  $\phi(\tau)$  are related by

$$(ma)^2 = \frac{1}{2\beta^2} \left( \beta^2 - 1 + \frac{1 + \beta^2}{\cosh \frac{2\phi}{M_{pl}}} \right) . \quad (2.17)$$

From (2.8),  $V(\phi)$  is written in  $a$  as

$$V(\phi) = \frac{m^2 M_{pl}^2}{2} \left[ 3(1 - \beta^2) + 4\beta^2 (ma)^2 \right] . \quad (2.18)$$

Finally, using (2.17) and (2.18), we obtain the form of scalar potential  $V(\phi)$  for our choice of  $f(a)$

$$V(\phi) = m^2 M_{pl}^2 \left( \frac{1 - \beta^2}{2} + \frac{1 + \beta^2}{\cosh \frac{2\phi}{M_{pl}}} \right) . \quad (2.19)$$

As advertised, this is exactly the potential appeared in the tachyon cosmology.

This potential has a maximum at  $\phi = 0$  with positive value

$$V(\phi = 0) = m^2 M_{pl}^2 \frac{3 + \beta^2}{2} > 0 , \quad (2.20)$$

and two vacua at  $\phi = \pm\infty$

$$V(\phi = \pm\infty) = m^2 M_{pl}^2 \frac{1 - \beta^2}{2} . \quad (2.21)$$

Note that the sign of vacuum energy at  $\phi = \pm\infty$  depends on the value of  $\beta$ . For definiteness we consider the case  $\beta = 1$ . Analysis of other cases is straightforward.

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<sup>2</sup>  $\text{sn}(z, k)$  is the inverse of the elliptic integral

$$\text{sn}^{-1}(z, k) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} . \quad (2.12)$$

### 2.1. $\beta = 1$ case

When  $\beta = 1$ , the scalar potential (2.19) becomes

$$V(\phi) = \frac{2m^2 M_{pl}^2}{\cosh \frac{2\phi}{M_{pl}}} . \quad (2.22)$$

$V(\phi)$  has an unstable tachyonic vacuum at  $\phi = 0$  and stable zero energy vacua at  $\phi = \pm\infty$ .

Near  $\phi = 0$ ,  $V(\phi)$  behaves as

$$V(\phi) \sim 2m^2 M_{pl}^2 - 4m^2 \phi^2, \quad (|\phi| \ll M_{pl}) . \quad (2.23)$$

From this we see that the parameter  $m$  sets the mass scale of the tachyon.

The solution for  $a(\tau)$  and  $\phi(\tau)$  for the  $\beta = 1$  case is

$$a(\tau) = \frac{1}{m} \text{sn}(m\tau, i), \quad \phi(\tau) = M_{pl} \tanh^{-1} \left( \frac{\text{cn}(m\tau, i)}{\text{dn}(m\tau, i)} \right) . \quad (2.24)$$

From the periodicity of Jacobi elliptic functions, we can easily see that

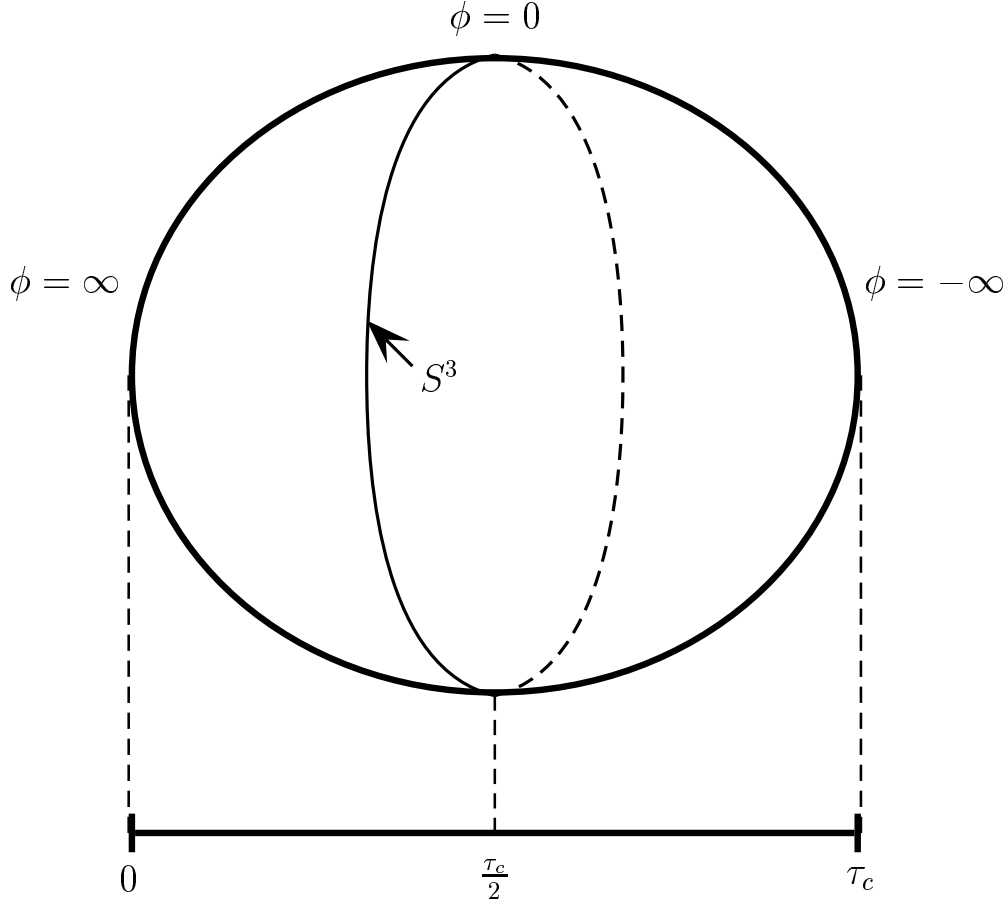
$$a(0) = 0, \quad a(\tau_c/2) = \frac{1}{m}, \quad a(\tau_c) = 0 , \quad (2.25)$$

$$\phi(0) = +\infty, \quad \phi(\tau_c/2) = 0, \quad \phi(\tau_c) = -\infty , \quad (2.26)$$

where  $\tau_c$  is given by

$$\tau_c = \frac{2}{m} \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2\pi}m} \Gamma\left(\frac{1}{4}\right)^2 . \quad (2.27)$$

Namely, this solution interpolates between two vacua at  $\phi = \pm\infty$ , and at each vacuum the radius of  $S^3$  shrinks to zero. Since  $a(\tau)$  behaves as  $a \sim \tau$  near  $\tau = 0$ , the metric (2.2) is smooth at  $\tau = 0$ . Also, the geometry is smooth at the other end  $\tau = \tau_c$ , since  $a(\tau) = a(\tau_c - \tau)$ . Therefore, our solution is an  $S^3$  bundle over the interval  $[0, \tau_c]$  with vanishing  $S^3$ 's at the boundaries. It is clear that the topology of the total space is  $S^4$  (see fig. 1). The maximum radius of  $S^3$  is achieved at the tachyonic vacuum  $\phi = 0$ , and its value is given by the inverse of tachyon mass:  $a(\tau_c/2) = 1/m$ .



**Fig. 1:** The instanton solution interpolating two vacua at  $\phi = \pm\infty$ .  $S^3$  is fibered over the interval  $\tau \in [0, \tau_c]$ . The topology of this solution is  $S^4$ .

### 3. Discussion

We found an exact instanton solution of tachyon cosmology. We emphasize that we did not use any approximation, such as a thin-wall approximation. Our solution can be used as a zero-th order term for a more interesting case, *i.e.* a bubble nucleation in the false vacuum

$$V(\infty) - V(-\infty) = \epsilon \neq 0, \quad (3.1)$$

assuming that  $\epsilon \ll V(0)$ . Also, it is interesting to consider the half of our Euclidean solution as the Hartle-Hawking no boundary state [8], and glue the Lorentzian inflationary spacetime at  $\tau = \tau_c/2$ . It would be also interesting to consider the relation to the S-brane solution [9].

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